Supporting Information for "Solid earth uplift due to contemporary ice melt above low-viscosity regions of the upper mantle"

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Introduction Texts S1 to S3 provide the mathematical description of the model. Table S1 gives the layer properties of the earth models used in our study.

Text S1: Conservation equations.

For the GIA simulations, ASPECT solves the 3-D incompressible conservation equations assuming an infinite Prandtl number, the Boussinesq approximation, and isothermal flow. The constitutive equations thus only consist of the momentum and continuity equation. Force terms are added on the right-hand-side of the momentum equation to account for the viscoelastic behavior and boundary traction. For incompressible and isothermal flow, under the Boussinesq approximation, the momentum equation (Eq. 1) and continuity equation (Eq. 2) reduce to:

$$-\nabla \cdot [2\eta_{\mathrm{eff}}\dot{\epsilon}(\mathbf{u})] + \nabla p = \rho\mathbf{g} + \nabla \cdot F_{\mathrm{e}} + \nabla \cdot F_{\mathrm{t}} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

where η_{eff} is the effective viscoelastic viscosity, $\dot{\epsilon}$ the strain rate, $\mathbf{u}$ the velocity, p the total pressure, ρ the density, $\mathbf{g}$ the gravity vector, F_{e} the elastic force term, F_{t} the boundary traction force term, and $2\eta_{\mathrm{eff}}\dot{\epsilon}(\mathbf{u})$ represents the deviatoric stress.

Text S2: Viscoelastic rheology.

The viscoelastic rheology is implemented through an elastic force term, and an effective viscosity in the momentum equation (Eq. 1) that accounts for the viscous and elastic deformation mechanisms. Our methodology for modeling viscoelasticity in ASPECT follows the approach of Moresi et al. (2003), as outlined in Sandiford et al. (2021). Within a timestep, first the viscoelastic stresses are updated, then the properties are updated through a reaction term in the advection equation, followed by constructing and solving the Stokes system with the elastic force term.

The velocity gradient tensor I^t and deviatoric stress tensor τ^t are constructed from the velocity solution and stored stress components of the previous timestep t. The new viscoelastic stresses become:

$$\tau^{t+\Delta t} = \eta_{eff}\left(2\hat{D}^t + \frac{\tau^t}{\mu\Delta t} + \frac{W^t\tau^t - \tau^t W^t}{\mu}\right) \tag{3}$$

with

$$\hat{D} = \frac{1}{2}\left(I + I^T\right) \tag{4}$$

$$W = \frac{1}{2}\left(I - I^T\right) \tag{5}$$

$$\eta_{eff} = \eta \frac{\Delta t}{\Delta t + \alpha} \tag{6}$$

where superscript t and $t + \Delta t$ indicate the previous and current timestep, and μ is the shear modulus. $\hat{D}$ and W are the deviatoric rate of deformation tensor (Eq. 4) and the spin tensor (Eq. 5), respectively, and are a function of the velocity gradient tensor. η_{eff} is the effective viscosity (Eq. 6) and is defined as a function of viscosity, timestep size and shear relaxation time α, where $\alpha = \eta/\mu$.

With the viscoelastic stresses of the previous and current timestep, the reaction term for the deviatoric stress q is determined. Materials are being tracked on compositional fields and for each field $c_i\left(x, t\right)$ with $i = 1...C$ an advection equation is solved which updates the stresses on the fields:

$$\frac{\partial c_i}{\partial t} + \mathbf{u} \cdot \nabla c_i = q_i \tag{7}$$

with

$$q = \tau^{t+\Delta t} - \tau^t \tag{8}$$

Then, the Stokes system (Eq. 1 and 2) is constructed with the updated deviatoric stress and the elastic force term, which is defined as:

$$F_e = -\frac{\eta_{eff}}{\eta_e} \tau^{t+\Delta t} \tag{9}$$

where η_e is the elastic viscosity and equals $\mu\Delta t$. With this Stokes system, we can solve for the new velocity field.

Text S3: Boundary conditions.

In this study, we use a free surface on the top boundary (Rose et al., 2017) (i.e. the earth's surface). The free surface is defined as having zero stress on the boundary. Thus, the following condition must be satisfied: $\sigma \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the vector normal to the boundary and total stress $\sigma = 2\eta_{\text{eff}}\dot{\epsilon}\,(\mathbf{u})$ (i.e. the total stress is the deviatoric stress since there is no pressure gradient at the top surface). When there is flow across the boundary, the mesh must be able to deform to satisfy the above condition. On the free surface, mesh velocity $\mathbf{u}_m$ is calculated as follows:

$$\mathbf{u}_m = (\mathbf{u} \cdot \mathbf{n})\,\mathbf{n} \tag{10}$$

Using this approach, the Eulerian advection terms need to be corrected for the mesh velocity. The momentum and continuity equations become:

$$-\nabla \cdot [2\eta_{\text{eff}}\dot{\epsilon}(\mathbf{u} - \mathbf{u}_m)] + \nabla p = \rho\mathbf{g} + \nabla \cdot F_{\text{e}} + \nabla \cdot F_{\text{t}} \tag{11}$$

$$\nabla \cdot (\mathbf{u} - \mathbf{u}_m) = 0 \tag{12}$$

Furthermore, a quasi-implicit integration scheme is used to prevent free surface position instabilities arising from small deviations in the free surface location.

Next to the free surface on the top boundary, we also apply a traction force to represent the surface ice loading. The ice loading is a known external force, resulting in an unknown velocity. The given pressure is applied as a force that is normal to the boundary. The boundary traction is represented as F_t in the momentum equation (Eq. 1). The other boundaries are either free-slip or open boundaries. The free-slip boundary requires that the flow is tangential to the boundary, i.e. $\mathbf{u} \cdot \mathbf{n} = 0$. For the open boundary, material can flow in and out of the model domain. Because inflow and outflow velocities are unknown, a 1D lithostatic pressure profile at a location of choice is computed from the model start situation and applied as boundary traction.

Table S1. Layer properties for case A. The density is the average mantle density and the shear moduli are PREM-averaged values for their depths (Dziewonski & Anderson, 1981).

Layer	Thickness T (km)	Density ρ (kg m^{-3})	Shear modulus μ (Pa)	Viscosity η (Pa s)	Relaxation time $\tau(=\eta/\mu)$ (yr)
Lithosphere, $_L$	45	4450	$0.45{\cdot}10^{11}$	$1{\cdot}10^{40}$	$7{\cdot}10^{21}$
Asthenosphere, $_A$	200	4450	$1.75{\cdot}10^{11}$	$1{\cdot}10^{19}$	1.8
Upper mantle, $_M$	255	4450	$1.75{\cdot}10^{11}$	$5{\cdot}10^{20}$	90.5