LECTURE 3: EARTH'S FIGURE, GRAVITY, AND GEOID

Earth's shape, tides, sea level, internal structure, and internal dynamics, are all controlled by gravitational forces. To understand gravitation and how it affects Earth, we start with Newton's laws:

Gravitational Potential

For a point mass:

Newton's law of gravitation: $\vec{F} = m\vec{a} = -G\frac{mM}{r^2}$

Then the acceleration due to gravity is: $g = -G\frac{M}{r^2}r$

The work by a force F on an object moving a distance dr in the direction of the force is: $dW = F dr$

The change in potential energy is: $dE_p = -dW = -F dr$

The gravitational potential U_G is the potential energy per unit mass in a gravitational field. Thus: $m dU_G = -F dr = -mg dr$

Then the gravitational acceleration is: $\vec{g} = -\vec{\nabla}U = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)U$

The gravitational potential is given by: $U_G = -G\frac{M}{r}$

For a distribution of mass:

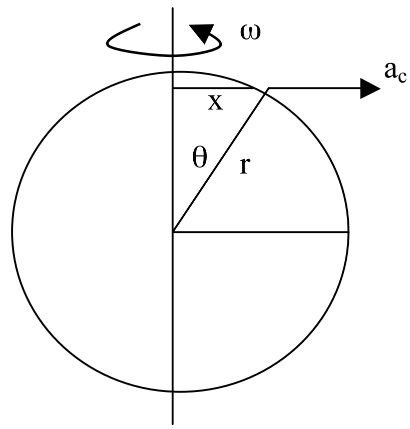
If a mass is distributed within a body of volume V, then we can integrate to find the total potential: $U_G = -G\int_V \frac{\rho(\vec{r})}{r} dV$

For the special case of a spherical shell of thickness t, applying this integral yields: $U_G = -\frac{GM}{r}$ as if the sphere were concentrated at the center.

Thus, everywhere outside a sphere of mass M: $U_G = -\frac{GM}{r}$

Centrifugal Potential

For a rotating body such as Earth, a portion of gravitational self-attraction drives a centripetal acceleration toward the center of the Earth. When viewed in the frame of the rotating body, the body experiences a centrifugal acceleration away from the Earth's axis of rotation.



Angular velocity: $\omega = \frac{d\theta}{dt} = \frac{v}{x}$ where $x = r\sin\theta$

Centrifugal acceleration: $a_c = \omega^2 x = \frac{v^2}{x}$

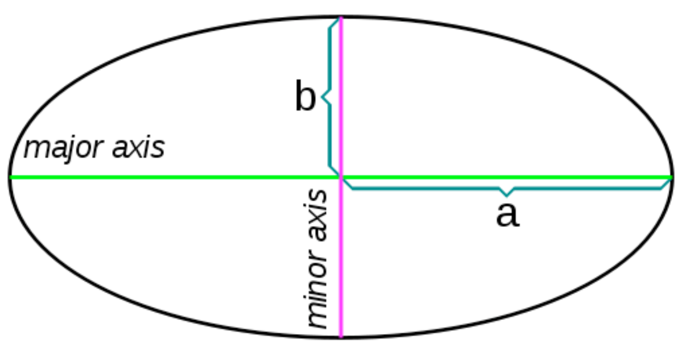
But $\vec{a}_c = -\vec{\nabla} U_c$, so we can calculate the centrifugal potential by integrating:

$$U_c = -\frac{1}{2}\omega^2 x^2 = -\frac{1}{2}\omega^2 r^2 \sin^2\theta$$

Figure of the Earth

Earth's actual surface is an equipotential surface (sea level), a surface for which $U_G + U_c =$ constant. The figure of the Earth a smooth surface that approximates this shape and upon which more complicated topography can be represented. The earth approximates an oblate spheroid, which means it is elliptically-shaped with a longer equatorial radius than a polar radius.

The flattening (or oblateness) is the ratio of the difference in radii to the equatorial radius:



$$f = \frac{a-b}{a}$$

For earth, *f*=0.00335287, or 1/298.252, and the difference in the polar and equatorial radii is about 21 km.

The **International Reference Ellipsoid** is an ellipsoid with dimensions:

Equatorial Radius:	$a = 6378.136$ km
Polar Radius	$c = 6356.751$ km
Radius of Equivalent Sphere:	$R = 6371.000$ km
Flattening	$f = 1/298.252$
Acceleration Ratio	$m = \frac{a_C}{a_G} = \frac{\omega^2 a^3}{GM_E} = 1/288.901$
Moment of Inertia Ratio	$H = \frac{C-A}{C} = 1/305.457$

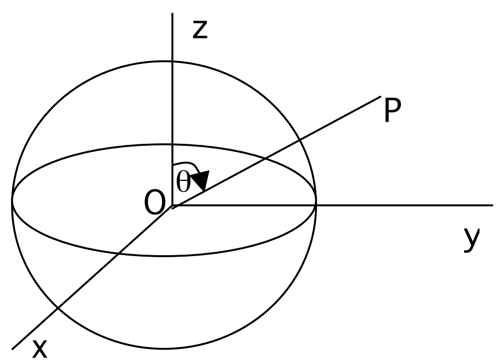
Hydrostatic equilibrium predicts that the flattening should be: f=1/299.7

This is smaller than the observed flattening by about 113 m [see Chambat et al., Flattening of the Earth: further from hydrostaticity than previously estimated, *Geophys. J. Int.*, 183, 727-732, 2010].

The gravitational potential of an ellipsoid is given by:

$$U_G = -G\frac{M_E}{r} - G\frac{(C-A)}{r^3}\frac{(3\cos^2\theta - 1)}{2} = -G\frac{M_E}{r} - G\frac{(C-A)}{r^3}P_2(\cos\theta)$$

where A and C are the moments of inertia about the equatorial and polar axes.



More generally:

$$U_G = -G\frac{M_E}{r}\left(1 - \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^2 J_n P_n(\cos\theta)\right)$$

Where P_n are the Legendre polynomials and the coefficients J_n are measured for Earth. The most important is the dynamical form factor: $J_2 = \frac{C-A}{M_E R^2} = 1082.6 \times 10^{-6}$

The next term, J_3, describes pear-shaped variations: a ~17 m bulge at North pole and ~7 m bulges at mid-southern latitudes (~1000 times smaller than J_2)

The gravitational potential of the Earth (the geopotential) is given by:

$$U_g = U_G - \frac{1}{2}\omega^2 r^2 \sin^2\theta = -\frac{GM}{r} + \frac{G}{r^3}(C-A)\left(\frac{3\cos^2\theta - 1}{2}\right) - \frac{1}{2}\omega^2 r^2 \sin^2\theta$$

The geopotential is a constant (U_0) everywhere on the reference ellipsoid. Then:

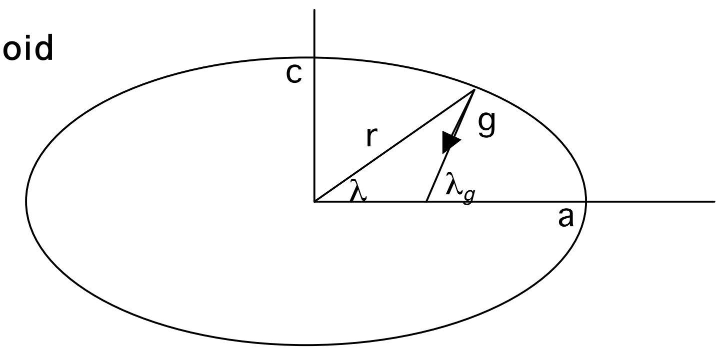
At the equator: $$U_0 = -\frac{GM}{a} + \frac{G}{2a^3}(C-A) - \frac{1}{2}\omega^2 a^2$$

At the pole: $$U_0 = -\frac{GM}{c} + \frac{G}{c^3}(C-A)$$

Then: $$f = \frac{a-c}{c} = \frac{(C-A)}{M_E a^2}\left(\frac{a^2}{c^2} + \frac{2c}{a}\right) + \frac{1}{2}\frac{a^2 c\omega^2}{GM_E} \approx \frac{3}{2}J_2 + \frac{1}{2}m$$

Where we have approximated $a \sim c$ on the right hand side.

Gravity on the Reference Ellipsoid



To first order: $r = a\left(1 - f\sin^2\lambda\right)$

Geocentric latitude = λ

(measured from center of mass)

Geographic latitude = λ_g

(in common use)

To first order: $\sin^2\lambda \approx \sin^2\lambda_g - f\sin^2 2\lambda_g$

The acceleration of gravity on the reference ellipsoid is given by: $\vec{g} = -\vec{\nabla} U_g$

Performing this differentiation gives: $$|g| = \frac{GM}{r^2} - \frac{3GM_E a^2 J_2}{r^2}\frac{3\sin^2\lambda - 1}{2} - \omega^2 r\cos^2\lambda$$

Rewriting and simplifying gives: $$g = g_e\left[1 + \left(2m - \frac{3}{2}J_2\right)\sin^2\lambda\right]$$

Writing in terms of λ_g gives: $$g = g_e\left[1 + \left(\frac{5}{2}m - f - \frac{17}{14}mf\right)\sin^2\lambda_g + \left(\frac{f^2}{8} - \frac{5}{8}mf\right)\sin^2 2\lambda_g\right]$$

$$g = 9.780327\left[1 + 0.0053024\sin^2\lambda_g + 0.0000059\sin^2 2\lambda_g\right]$$

Equatorial gravity is: $$g_e = \frac{GM}{a^2}\left[1 - \frac{3}{2}J_2 - m\right] = 9.780327 \text{ m/s}^2$$

This allows us to compute the polar gravity: $g_p = 9.832186 \text{ m/s}^2$

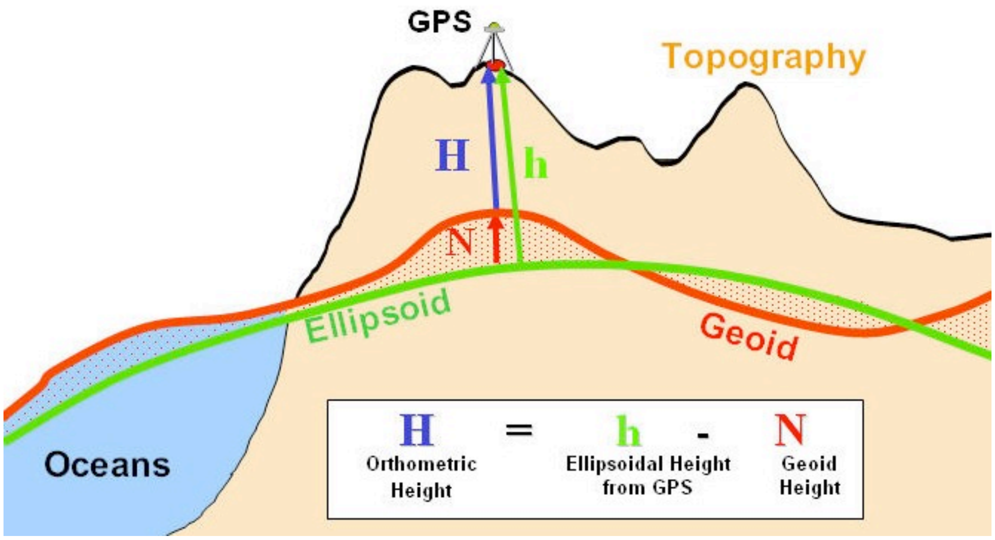
The poleward increase in gravity is 5186 mgal, and thus only about 0.5% of the absolute value (gravity is typically measured in units of mgal = 10^{-5} m/s^2). Gravity decreases toward to pole because the pole:

(1) is closer to the center of Earth than the equator (6600 mgal)

(2) does not experience centrifugal acceleration (3375 mgal)

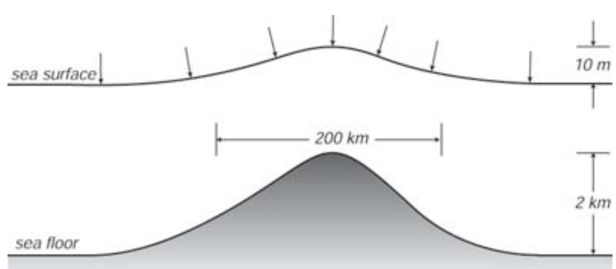
But the equator has more mass (because of the bulge), which increases the equatorial gravity. Together these three affects yield the 5186 mgal difference.

Earth's Geoid

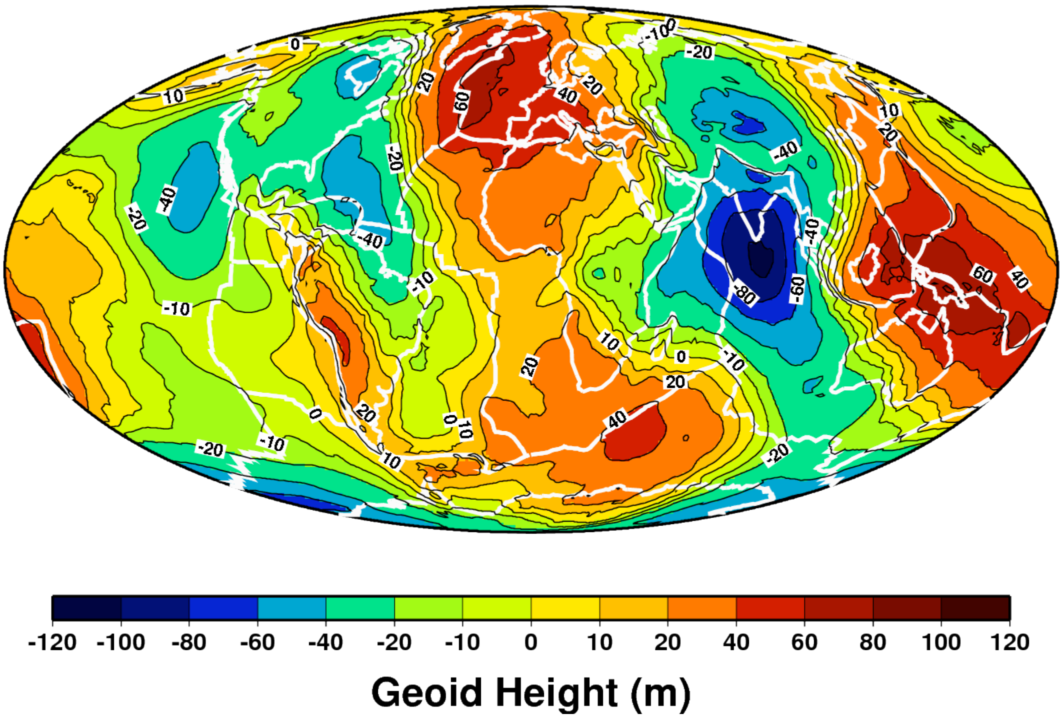


The **geoid** is the equipotential surface that defines sea level, and is expressed relative to the reference ellipsoid. Temporal variations in the geoid are caused by lateral variations in the internal densities of the Earth, and by the distribution of masses (primarily hydrological) upon the surface of the Earth.

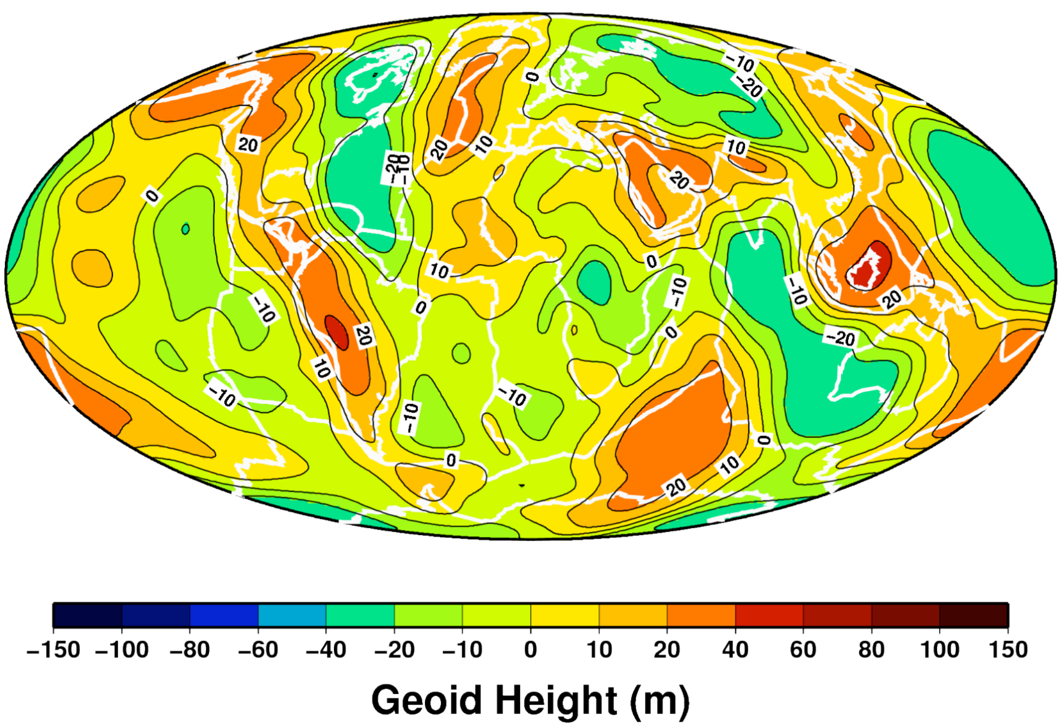
Mass excess (either subsurface excess density or positive topography) deflects the geoid upwards.



Observed Geoid (EGM96)

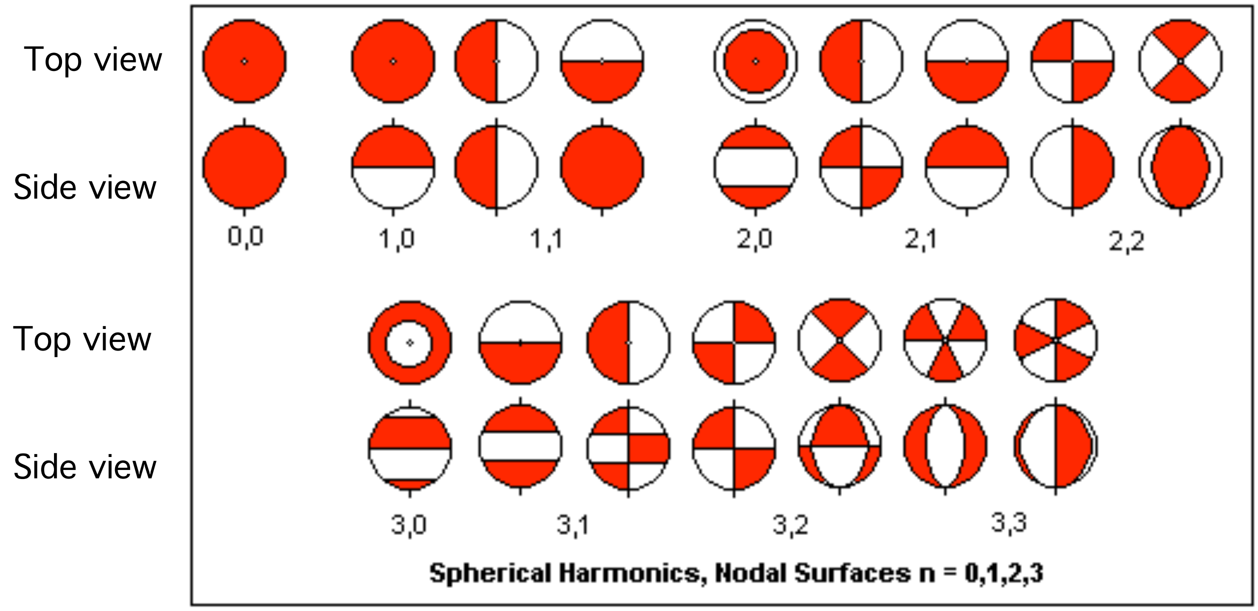


Observed Geoid (EGM96, degrees 4–25)



Spherical Harmonics

The geoid (and any function on a sphere) can be expressed in terms of spherical harmonics of degree n and order m: $Y_n^m = \left(a_n^m \cos m\phi + b_n^m \sin m\phi\right) P_n^m\left(\cos\theta\right)$



The power spectrum of the geoid is given by:

$$P_n = \sum_{m=0}^{n}\left(a_{nm}^2 + b_{nm}^2\right)$$

The dominance of the low-harmonic degrees in the geoid power spectrum indicate that the dominant shape of the geoid is controlled by structures deep within the mantle.