

Geoid, Topography and Distribution of Landforms

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1. GRAVITATION POTENTIAL

1.1. Spherical Harmonic Expansion of the Earth Gravitational Potential. Stokes Coefficients

The gravitational potential at point P outside the Earth due to the heterogeneous mass distribution inside the Earth volume is

$$U = G \int_M \frac{dM}{a} \quad (1)$$

where G is gravitational constant, M is Earth's mass and a is distance between a mass element dM and point P. The potential U is conveniently expressed through a spherical harmonic expansion in a terrestrial reference frame [1,2]

$$U = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^n \left(\frac{R}{r}\right)^n \times (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm}(\sin \varphi) \quad (2)$$

where R is equatorial radius, r, φ , λ are spherical coordinates of point P. $P_{nm}(\sin \varphi)$ is Legendre polynomial of degree n and order m. C_{nm} and S_{nm} refer to the Stokes' coefficients which represent integral functions of the mass distribution inside the Earth [1, 2]

$$\left. \begin{aligned} C_{nm} \\ S_{nm} \end{aligned} \right\} = \frac{1}{MR^n} (2 - \delta_{0n}) \frac{(n-m)!}{(n+m)!} \times \int_M r'^n P_{nm}(\sin \varphi') \begin{Bmatrix} \cos m\lambda' \\ \sin m\lambda' \end{Bmatrix} dM \quad (3)$$

With definition (3), $C_{00} = 1$ and $S_{00} = 0$.

It is classical to choose the coordinate system located at the Earth center of mass, hence C_{10} , C_{11} , $S_{11} = 0$.

The second degree Stokes' coefficients are related to the moments and products of inertia I_{ij} with respect to the axes of the reference system

$$C_{20} = \frac{-1}{MR^2} \left[I_{33} - \frac{1}{2} (I_{11} + I_{22}) \right] \quad (4)$$

$$C_{21} = \frac{I_{13}}{MR^2} \quad S_{21} = \frac{I_{23}}{MR^2} \quad (5)$$

$$C_{22} = \frac{1}{4MR^2} (I_{22} - I_{11}) \quad S_{22} = \frac{I_{12}}{2MR^2} \quad (6)$$

The choice of the coordinate system is usually made with the assumption that the z-axis lies close to the mean axis of rotation and mean axis of maximum inertia. Since the latter two are close together, C_{21} and S_{21} are small quantities. C_{20} is the largest of all Stokes' coefficients. It is called the dynamical flattening. It is on the order of 10^{-3} . All other coefficients are on the order of 10^{-6} .

Stokes' coefficients are classically derived from the analysis of orbital perturbations of Earth' satellites. Sets of C_{nm} , S_{nm} coefficients are improved regularly. Table 1 gives the first C_{nm} , S_{nm} coefficients (up to degree 6) of the GEM-T3 geopotential model [3]. The coefficients are

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normalized, i.e., are multiplied by a normalization factor equal to

$$\frac{1}{(2n + 1)^{1/2}} \text{ for } n = 0 \quad (7)$$

and

$$\left[\frac{(n + m)!}{2(2n + 1)(n - m)!} \right]^{1/2} \text{ for } n \neq 0 \quad (8)$$

The GEM-T3 model, complete to degree and order 50, is derived from tracking data of 31 satellites and combines satellite altimeter data over oceans and surface gravimetric data. Other combined geopotential models have been derived up to degree 360 [4].

The long-wavelength geoid surface (equipotential surface of the Earth gravity field coinciding with the mean sea level) is presented in Figure 1. It is based on the C_{nm} , S_{nm} coefficients of the GEM-T3 model up to degree 50.

During the past decade, shorter wavelength geoid undulations have been mapped directly by altimetric satellites over the whole oceanic domain. Figure 2 shows the medium and short-wavelength geoid undulations mapped by the Geosat satellite. Geoid undulations are due to density heterogeneities in the mantle. At the shortest wavelengths, geoid undulations result from topography and crustal density variations.

1.2. Power Spectrum of the Geopotential

The power spectrum or degree variance of the geopotential is given by

$$P_n = \sum_{m=0}^n (C_{nm}^2 + S_{nm}^2) \quad (9)$$

Figure 3 shows a Log plot of the power spectrum of a recent geopotential model as well as of the Kaula's empirical rule [5] stating that the dimensionless power spectrum of the geopotential follows as

$$P_n \approx (2n + 1) (10^{-5} n^{-2})^2 \quad (10)$$

2. GEOID AND GRAVITY ANOMALIES

2.1. Geoid, Geoid Height, Gravity Anomalies and Deflection of the Vertical

The geoid is defined as the equipotential surface of the gravity potential $W = U + Z$ and coincides with the mean sea level. Z is the kinetic potential due to the rotational motion of the Earth

TABLE 1. Spherical Harmonic Normalized Coefficients of the GEM-T3 Geopotential Model (units of 10^{-6})

Index		Value	
n	m	C	S
1	0		
1	1		
2	0	-484.1651	
2	1		
2	2	2.439	-1.400
3	0	0.9572	
3	1	2.0277	0.2492
3	2	0.9045	-0.6194
3	3	0.7203	1.4139
4	0	0.5395	
4	1	-0.5361	-0.4734
4	2	0.3502	0.6630
4	3	0.9909	-0.2009
4	4	-0.1888	0.3094
5	0	0.0683	
5	1	-0.0582	-0.0960
5	2	0.6527	-0.3239
5	3	-0.4523	-0.2153
5	4	-0.2956	0.0497
5	5	0.1738	-0.6689
6	0	-0.1495	
6	1	-0.0769	0.0269
6	2	0.0487	-0.3740
6	3	0.0572	0.0094
6	4	-0.0868	-0.4713
6	5	-0.2673	-0.5367
6	6	0.0096	-0.2371
GM		0.398600436 x 10 ¹⁵ m ³ s ⁻²	
Equatorial Radius		0.6378137 x 10 ⁷ m	
Flattening		1/298.257	
Light Velocity		0.299792458 x 10 ⁹ m s ⁻¹	

$$Z = \frac{1}{2} \omega^2 r^2 \cos^2 \varphi \quad (11)$$

ω is speed of rotation, r is geocentric distance and φ is latitude.

Geoid height is measured above a reference surface, a conventional ellipsoid of revolution approximating the real figure of the Earth. The reference ellipsoid is usually

Geoid Surface Computed from GEM-T3

Contour Interval = 5 m

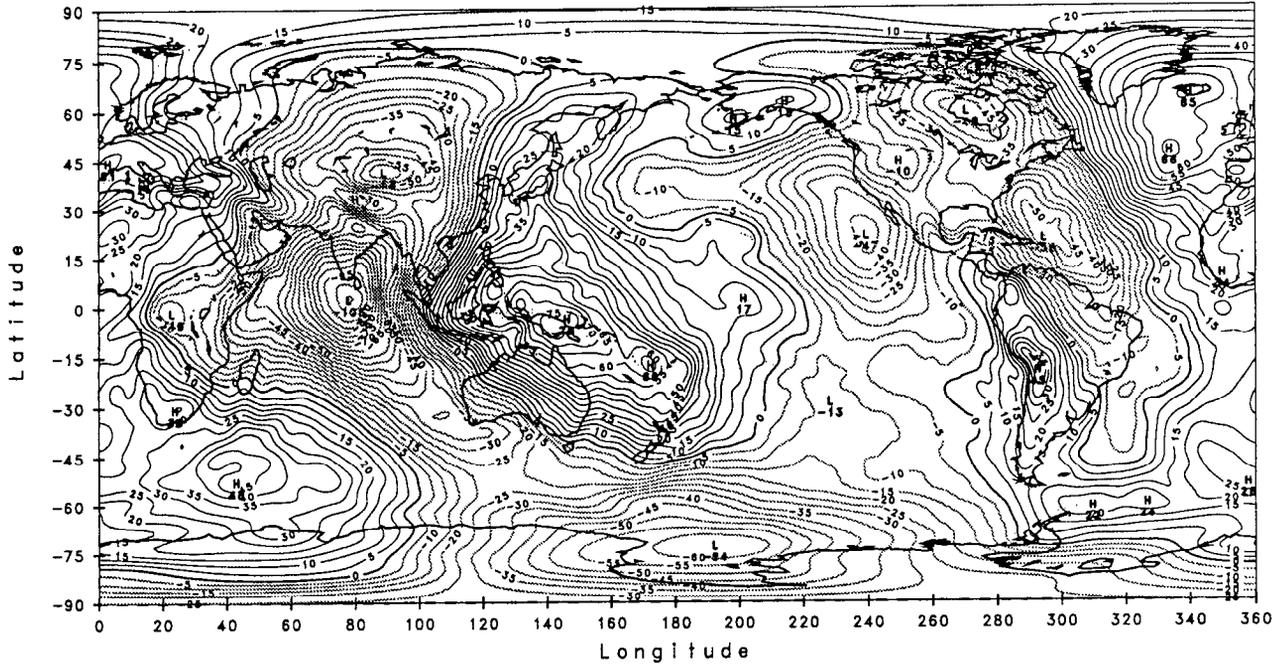


Fig. 1 : Long-wavelength geoid surface deduced from the C_{nm} , S_{nm} (up to $n = 50$) coefficients of the geopotential (GEM-T3 model, [3]). The C_{nm} , S_{nm} coefficients are derived from the analysis of orbital perturbations of a large number of geodetic satellites in different inclinations (Level curves at 5 m interval). Solid and dashed curves correspond respectively to positive and negative geoid anomalies.

defined as the equipotential surface $V = \text{constante}$ with

$$V = \frac{GM}{r} \left[1 + \sum_{n=1}^{\infty} C_{2n0} \left(\frac{R}{r} \right)^{2n} P_{2n0}(\sin \varphi) \right] + Z \quad (12)$$

The reference ellipsoid has same potential, same mass and center of mass, same angular velocity and same dynamical flattening as the real Earth. The geoid height measured above the reference ellipsoid (Figure 4) is related to the perturbing potential T , difference between true potential W and potential of the reference ellipsoid V (at the same point), through

$$N = \frac{T}{g} \quad (13)$$

g is mean surface gravity
The actual gravity vector is

$$g = \text{grad } W \quad (14)$$

The gravity perturbation vector on the geoid is given by

$$\delta g_p = g_p - \gamma_p \quad (15)$$

where $\gamma = \text{grad } V$ is the normal gravity vector of the ellipsoid.
A more useful quantity is the gravity anomaly vector defined as

$$\Delta g_p = g_p - \gamma_q$$

where points p and q are on the geoid and ellipsoid respectively.
The magnitude and direction of the vector Δg are called respectively gravity anomaly and deflection of the vertical (i.e., angle between the normal to the geoid and the normal to the ellipsoid). Δg is related to the perturbing potential through

$$\Delta g = - \left(2 \frac{T}{r} + \frac{\partial T}{\partial r} \right) \quad (16)$$

The deflection of the vertical has two components (north-south ξ and east-west η). The relationships between components ξ and η of the deflection of the vertical, and

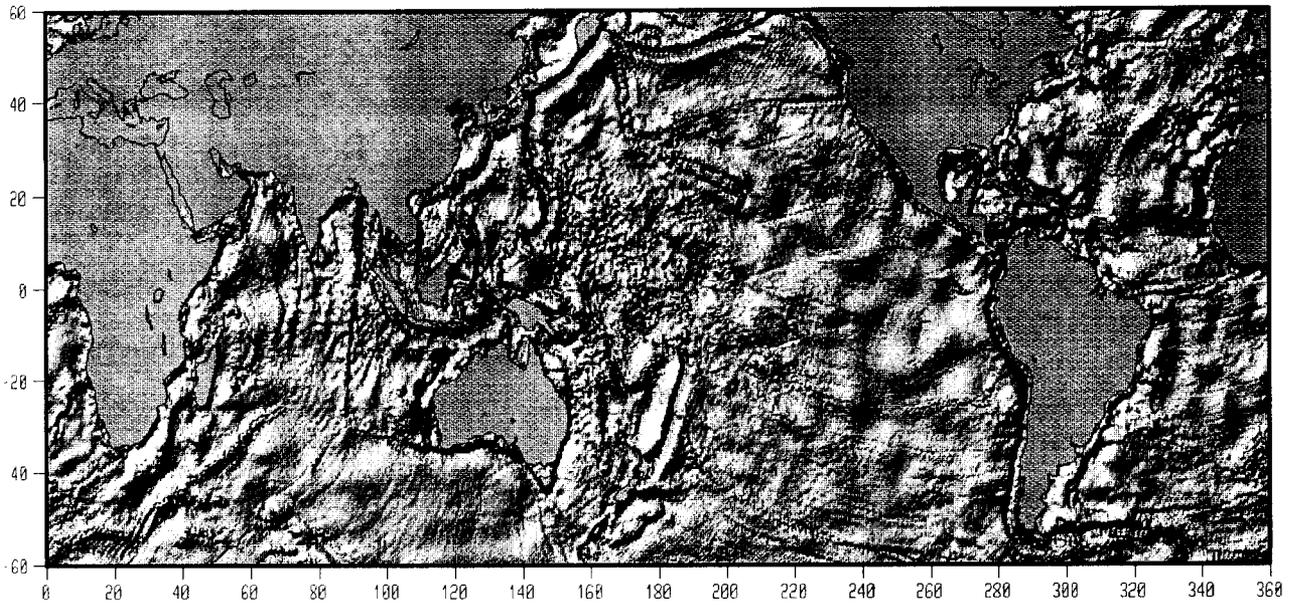


Fig. 2 : Marine geoid mapped by satellite altimetry. Since 1975, several geodetic satellites carrying onboard altimeter instruments have performed a direct mapping of sea surface heights hence of geoid undulations. The geoid surface presented above is derived from altimetric measurements collected in 1987 and 1988 by the Geosat satellite from the US Defense Mapping Agency. Owing to the dense coverage of oceanic areas, satellite altimetry has revealed high resolution features in the geoid surface.

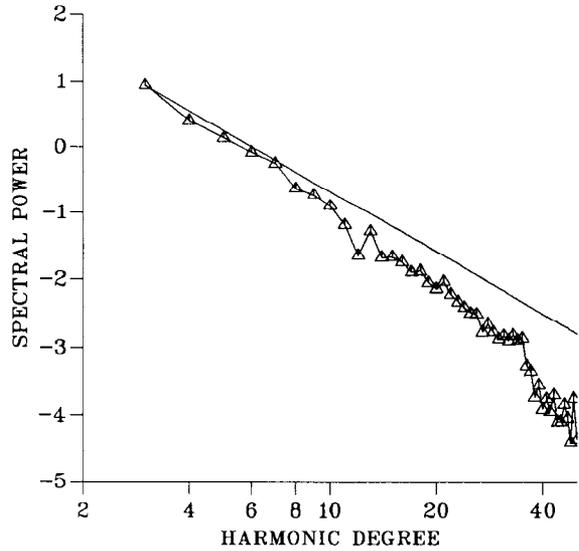


Fig. 3 : Log plot of the power spectrum of the GEM-T3 geopotential model (Δ) and of the Kaula's empirical rule (solid curve) versus harmonic degree.

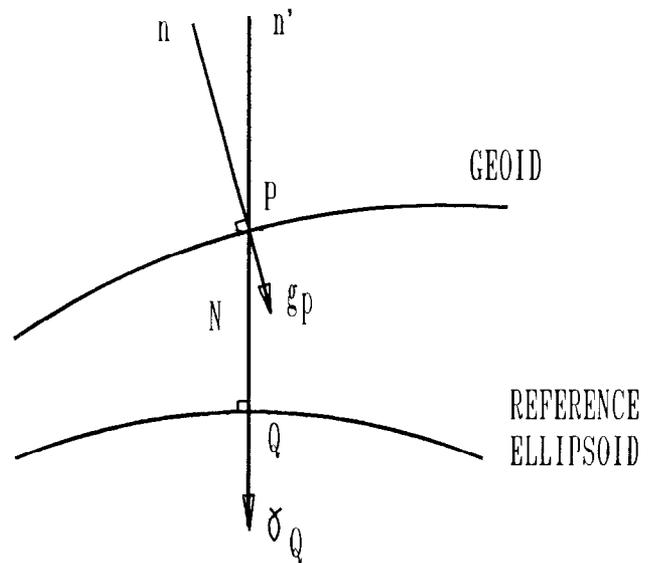


Fig. 4 : Schematic representation of the reference surfaces.

geoid height and gravity anomaly are

$$\xi = -\frac{1}{r} \frac{\partial N}{\partial \varphi} \quad (17)$$

$$\eta = -\frac{1}{r \cos \varphi} \frac{\partial N}{\partial \lambda} \quad (18)$$

$$-G \left[\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right] = \frac{\partial}{\partial r} \Delta g \quad (19)$$

x, y are local rectangular coordinates ($dx = R d\varphi$,
 $dy = R \cos \varphi d\lambda$)

For geophysical purposes, it is sometimes convenient to refer geoid height with respect to the hydrostatic reference ellipsoid, i.e., the ellipsoid which has the hydrostatic flattening of a fluid rotating Earth.

Table 2 gives the numerical values of Earth geodetic parameters adopted by the International Earth Rotation Service (IERS) standards [6].

2.2. Isostasy

The principle of isostasy states that topographic masses are balanced (or compensated) by underlying mass deficiency in such a way that at a given depth (the depth of compensation) the pressure is hydrostatic. To a good approximation, the Earth is in isostatic equilibrium. Isostatic compensation may be achieved through a variety of mechanisms. Most topographic loads of wavelengths < 50 km are supported by the strength of the lithosphere and are uncompensated. In the range 50-500 km, loads are supported by elastic flexure of the upper lithosphere. At wavelengths > 500 km, topography is in local isostatic equilibrium or dynamically supported [7, 8].

Classical models of local isostatic compensation are Airy and Pratt models (Figure 5). The Airy model assumes that the topography is balanced by a crust of constant density but variable thickness according to

$$t = \frac{\rho^* h}{(\rho_m - \rho_c)} \quad (20)$$

with $\rho^* = \rho_c$ for topography above sea level and

$\rho^* = \rho_c - \rho_w$ for topography below sea level.

t is crustal root, h is topographic height above or below sea level, ρ_w , ρ_c and ρ_m are seawater, crust and mantle densities.

In the Pratt model, the topography is compensated by lateral density variations in a layer of constant thickness above the depth of compensation. The variable density is

$$\rho = \frac{D^*}{H + h} \quad (21)$$

TABLE 2. Numerical Values of Earth's Geodetic Parameters According to the International Earth Rotation Service (IERS) Standards [6]

Name	Value in SI unit
Earth Mass (M)	5.973×10^{24} kg
Product of the Earth's mass by the gravitational constant (GM)	$3.986004418 \times 10^{14}$ m ³ s ⁻²
Observed dynamical flattening	
Normalized value (C ₂₀)	-484.1651×10^{-6}
Non Normalized value (J ₂)	1082.6362×10^{-6}
Equatorial radius (R _e)	6378136.3 m
Polar radius (R _p)	6356753.0 m
Angular velocity (ω)	7.292115×10^{-5} rad s ⁻¹
Observed inverse flattening (f^{-1})	298.257
Hydrostatic inverse flattening (f_h^{-1})	299.638
Equatorial gravity (g_e)	9.780328 m s ⁻²
Polar gravity (g_p)	9.832186 m s ⁻²
Hydrostatic dynamical flattening (C _{20h})	-480.2×10^{-6}
Equatorial moments of inertia (I ₁₁)	8.0094×10^{37} kg m ²
(I ₂₂)	8.0096×10^{37} kg m ²
Polar moment of inertia (I ₃₃)	8.0358×10^{37} kg m ²

with $D^* = \rho_c H$ for topography above sea level and $D^* = \rho_c H + \rho_w h$ for topography below sea level. H is the compensation depth.

$\rho_w = 1.03 \times 10^3$ kg m⁻³, $\rho_c = 2.8 \times 10^3$ kg m⁻³ and $\rho_m = 3.3 \times 10^3$ kg m⁻³,

For a two-dimensional locally compensated topography, the geoid height N is given by [7]

$$N = -\frac{2\pi G}{g} \int_0^H z \Delta \rho(x, z) dz \quad (22)$$

z is depth positive downward, $\Delta \rho(x, z)$ is 2-D density contrast occurring between $z = 0$ and $z = H$.

3. TOPOGRAPHY AND DISTRIBUTION OF LANDFORMS

3.1. Actual Earth Topography

The Earth topography presents a bimodal distribution with a peak near 0.5 km corresponding to the mean

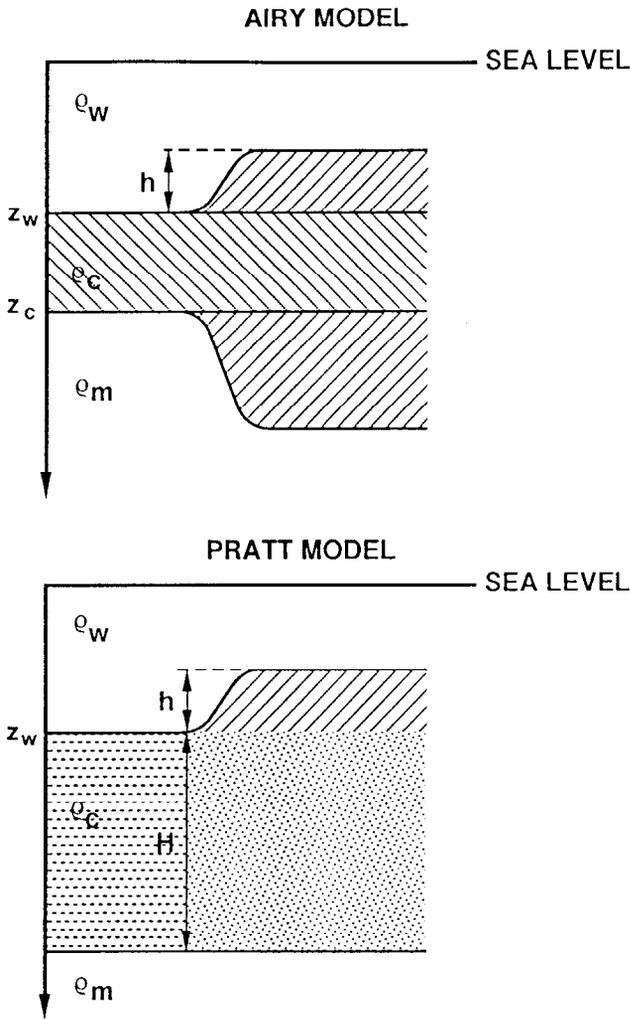


Fig. 5 : Classical models (Airy and Pratt) of local isostasy.

elevation of continental areas and a peak near 4.5 km corresponding to the mean ocean depth (Figure 6). Table 3 gives the coefficients t'_{nm} , t''_{nm} (up to degree 6) of the spherical harmonic expansion of the Earth topography T . Coefficient are defined according to

$$T(\varphi, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^n (t'_{nm} \cos m\lambda + t''_{nm} \sin m\lambda) \times P_{nm}(\sin \varphi) \quad (23)$$

The data set used to derive the t'_{nm} , t''_{nm} coefficients is the ETOPO 5 data base distributed by the National Geophysical Data Center in Boulder, Colorado. ETOPO 5 gives interpolated topography above and below sea level on a

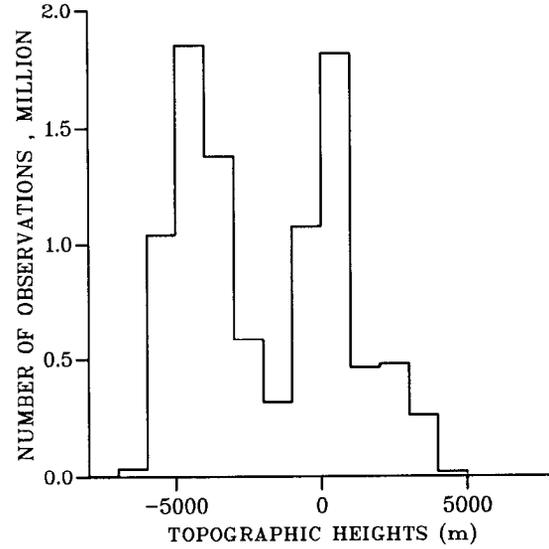


Fig. 6 : Histogram of the Earth topographic heights (above and below sea level) referred to sea level.

regular 5' x 5' grid. Other topographic data bases have also been developed (e.g., the OSUJAN89, [9]).

Figure 7 presents the power spectrum of T (defined as in relation (9)) showing a regular decrease with increasing degree. Peaks at $n=1$ and 3 relate to the preferential grouping of continents over a single hemispheric cap and in northern latitudes.

3.2. Oceanfloor Topography

The mean depth of ocean ridges is ≈ 2.5 km although regional variations of ± 1 km around the mean are observed. Depth of the oceanfloor increases regularly away from mid-ocean ridges as a result of thermal cooling and contraction of the oceanic lithosphere. Thermal subsidence of the seafloor is well approximated by an empirical relationship of the form [7]

$$d = d_0 + A t^{1/2} \quad (24)$$

d is seafloor depth referred to sea-level and positive downward, d_0 is mean depth of mid-ocean ridges and t is crustal age.

The value of A is around $350 \text{ m}/(\text{my})^{1/2}$ if d and d_0 are expressed in m and t in my.

Depth anomalies refer to oceanfloor topography corrected for thermal subsidence.

3.3. The Ocean-Continent Distribution

The uneven ocean-continent distribution is usually expressed through the ocean-continent function C , equal to 1 over oceanic areas and 0 over continents. It can be expressed through a spherical harmonic expansion

TABLE 3. Spherical Harmonic Normalized Coefficients of the Earth's Topography (in 10³m)

n	Index		Value	
	m	C	S	
0	0	-2.3890		
1	0	0.6605		
1	1	0.6072	0.4062	
2	0	0.5644		
2	1	0.3333	0.3173	
2	2	0.4208	0.0839	
3	0	-0.1683		
3	1	-0.1518	0.1244	
3	2	0.4477	0.4589	
3	3	0.1299	0.5733	
4	0	0.3612		
4	1	-0.2241	-0.2563	
4	2	-0.3928	0.0716	
4	3	0.3761	-0.1291	
4	4	-0.6387	0.4703	
5	0	-0.5514		
5	1	-0.0406	-0.0770	
5	2	-0.0216	-0.1577	
5	3	0.1232	0.0386	
5	4	0.5254	-0.0654	
5	5	-0.0549	0.2276	
6	0	0.2567		
6	1	0.0013	-0.0171	
6	2	0.0247	-0.1323	
6	3	0.0601	0.1865	
6	4	0.1960	-0.1737	
6	5	-0.1076	-0.2075	
6	6	0.0354	0.0282	

TABLE 4. Spherical Harmonic Normalized Coefficients of the Ocean-Continent Function (units of 10⁻¹)

n	Index		Value	
	m	C	S	
0	0	-7.102		
1	0	-1.193		
1	1	-1.076	-0.5905	
2	0	-0.5906		
2	1	-0.3954	-0.0061	
2	2	0.3861	-0.0117	
3	0	0.4747		
3	1	0.4709	-0.3871	
3	2	0.6450	-0.8860	
3	3	-0.1058	-0.8214	
4	0	-0.2791		
4	1	0.3633	0.2661	
4	2	0.8801	-0.2185	
4	3	-0.4984	0.0526	
4	4	0.1571	-1.0180	
5	0	1.0320		
5	1	-0.0379	0.1281	
5	2	0.4983	0.2619	
5	3	-0.3027	-0.1275	
5	4	-0.9289	0.3000	
5	5	0.0058	-0.4907	
6	0	-0.3264		
6	1	0.0612	0.1712	
6	2	0.1454	-0.0220	
6	3	0.0129	-0.2713	
6	4	-0.2704	0.2420	
6	5	0.1984	0.2467	
6	6	-0.0623	-0.1399	

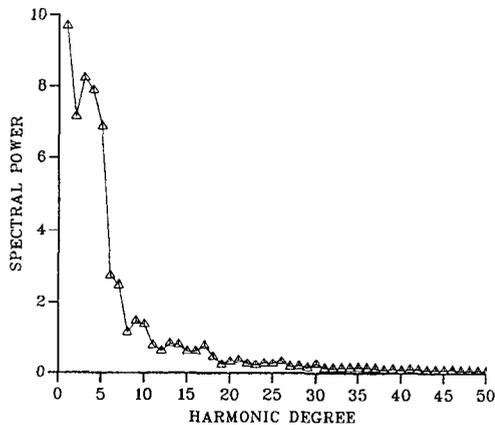


Fig. 7 : Power spectrum of the Earth topography (units : 10⁵ m²) versus harmonic degree.

$$C = \sum_{n=0}^{\infty} \sum_{m=0}^n (C_{onn} \cos m\lambda + S_{onn} \sin m\lambda)$$

$$\times P_{nm}(\sin \varphi) \tag{25}$$

Values of C_{onn}, S_{onn} are listed in Table 4 up to degree 6. The data set used to derive the ocean-continent function is the ETOPO 5 topography data base.

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